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SPACES HAVING UNIFORMITIES WITH LINEARLY ORDERED BASES

In this paper we consider topological spaces which have uniformities with linearly ordered bases (shortly, with l.o. bases). The main result asserts that each such a space X either is metrizable or $\dim X = 0$. A more detailed description of non-metrizable spaces with l.o. bases is given in Theorem 4.

There is used a covering treatment of uniformities such as in Isbell's book [2] (see also [1], p. 315).

If U is a uniformity on a space X and P_1 and P_2 are coverings belonging to U then $P_1 \succ P_2$ meanas that P_1 is a refinement of P_2 and $P_1 \succ_* P_2$ means that P_1 is a star refinement of P_2.

If a base B of U is lineary ordered, then B contains a cofinal and well-ordered subfamily which is also a base for U. By the weight of B we mean the least cardinal number which is a power of such a subfamily. If the weight of B is τ, then there exists a cofinal subfamily in B of type ω_τ (the least ordinal number of the power τ). Without loss of generality, we may assume that if the weight of B is τ then B is of the type ω_τ. Therefore, if the weight of B is τ and $B' \subset B$ is of the power less than τ, then there exists $P \in B$ such that $P \succ P'$ for each $P' \in B'$. In particular, if B is uncountable, then there exists such a P for each countable subfamily of B.

Recall that the countability of B means that the space X is metrizable.

LEMMA 1. *Let space X has a uniformity with l.o. base with the weight $\geqslant \tau$. Then, for every family F of open coverings of the power less than τ, there exists an open covering P such that $P \succ G$ for each $G \in F$.*

Proof. Let $x \in X$. For each $G \in F$ there exists $P_G \in B$ such that st $(x, P_G) \subset u$ for some $u \in G$. Let $P(x) \in B$ be such that $P(x) \succ P_G$ for each $G \in F$ (such a $P(x)$ a exists, since card $F < \tau$). We have st $(x, P(x)) \subset$

$\subset st(x, P_G) \subset u \in G$ for each $G \in F$ and some $u \in G$. We define $P =$ $= \{st(x, P(x)) : x \in X\}$.

THEOREM 1. *If a space X has a uniformity with l.o. base of the weight τ, then every family R of open sabsets of X with the cardinality $< \tau$ has an open interesection.*

Proof. Let $x \in \bigcap R$. For each $G \in R$ there exists $P_G \in B$ such that $st(x, P_G) \subset G$. Let $P \in B$ be such that $P \succ P_G$ for every $G \in R$. Thus $st(x, P) \subset st(x, P_G) \subset G$ for every $G \in R$ and therefore $st(x, P) \subset \bigcap R$. Thus $\bigcap R$ is open.

COROLLARY 1. (Kelley [3], exercise G, p. 272) If a space X has a uniformity with l.o. base, then it is metrizabe or every G_δ subset of X is open in X.

Proof. In fact, if l.o. base B is countable, then X is metrizable, and if B is uncountable, then each G_δ subset is open in X in virtue of Theorem 1.

COROLLARY 2. If a metrizable space has a uniformity with *non--metrizable l.o. base,* then it is discrete.

Proof. This follows from fact that every point in metrizable space is a G_δ set.

THEOREM 2. *Every space X having a uniformity with non-metrizable l.o. base has covering dimension equal to zero, i.e. dim $X = 0$.*

Proof. Let $\{V_i\}_{i=1}^{k}$ be a finite functionally open covering of space X. From theorem 5 in [1], p. 267, it follows that there exists a functionally closed covering $\{F_i\}_{i=1}^{k}$ such that $F_i \subset V_i$ for $i = 1, 2, \ldots, k$. Every F_i is G_δ set, so we conclude from Corollary 1 that F_i is closed-open set in X. Define $U_1 = F_1$ and $U_i = F_i \setminus \bigcup_{j=1}^{i-1} U_j$ for $i = 2, \ldots, k$. Let us notice that $\{U_i\}_{i=1}^{k}$ is an open covering of X, $U_i \subset V_i$ and $U_i \cap U_j = \emptyset$ for $i \neq j$, $i, j = 1, 2, \ldots, k$. Thus dim $X = 0$.

THEOREM 3. *Every topological space having a uniformity with l.o. base is paracompact.*

Proof. Let X be a topological space and let $B = \{P_\alpha : \alpha < \omega_\tau^0\}$ be a well-ordered l.o. base on X. Let P be any open covering of X. We define $V = \{st(x, P_{\alpha+2}) : st(x, P_\alpha) \subset u, u \in P, x \in X\}$. To prove that X is paracompact it suffices to show that $V \succ_* P$.

Let $v = st(x_0, P_{\alpha+2})$ be an element from V. We define β to be the smallest number such that $v \cap v' \neq \emptyset$ for some $v' = st(x', P'_{\beta+2})$ and $st(x', P_\beta) \subset u$ for some $u \in P$. Let us notice that $st(v, V) \subset \bigcup \{st(x, P_{\beta+2})$ $st(x, P_{\beta+2}) \cap st(x_0, P_{\beta+2}) \neq \emptyset\}$. Now we show that $st(v, V) \subset u$. It suffices to prove that if $st(x, P_{\beta+2}) \cap st(x_0, P_{\beta+2}) \neq \emptyset$ then $st(x, P_{\beta+2}) \subset u$.

There exist $u_{\beta+2} \in P_{\beta+2}$ and $u'_{\beta+2} \in P_{\beta+2}$ such that $x_0 \in u_{\beta+2}$, $x' \in st(u_{\beta+2}, P_{\beta+2}) \subset u_{\beta+1} \in P_{\beta+1}$, $x \in u'_{\beta+2}$ and $x_0 \in st(u'_{\beta+2}, P_{\beta+2}) \subset u'_{\beta+1} \in P_{\beta+1}$. Since $x' \in u_{\beta+1}$ and $x_0 \in u_{\beta+1} \cap u'_{\beta+1}$, we have $st(x, P_{\beta+1}) \subset st(u'_{\beta+1}, P_{\beta+1}) \subset u_\beta \in P_\beta$. where $x' \in u_\beta$. Thus $st(x, P_{\beta+2}) \subset st(x, P_\beta) \subset u$

From Theorems 2 and 3 and from Theorem 9 [1], 269, we get

COROLLARY 1. If a space X has a uniformity with *non-metrizable l.o. base*, then $\dim X = \operatorname{Ind} X = \operatorname{ind} X = 0$.

LEMMA 2. *If a space has a uniformity with uncountable l.o. base, then it has a uniformity with a l.o. base consiting of coverings of order* 1.

Proof. Let $\{P_\alpha : \alpha < \omega_\tau\}$ be a l.o. base on X of the weight τ. Now we define a zero domensional l.o. base $\{Q_\alpha : \alpha \leqslant \omega_\tau\}$. From Theorem 3 and from Theorem 3 in [1], p. 278, it follows that there exists an open covering Q_1 of order 1 and such that $Q_1 \succ P_1$. Let us assume that Q_α for $\alpha \leqslant \gamma$ are already defined. From Lemma 1 results the existence of a covering $P \succ P_\gamma$ and $P \succ Q_\alpha$ for $\alpha < \gamma$. Let Q_γ be such that $Q_\gamma \succ P$ and order of Q_γ is 1.

THEOREM 4. *If a space X has a uniformity with an uncountable l.o. base of the weight τ, then X is homeomorphic with the inverse limit* $\varprojlim$ $\{X_\alpha, \pi^\alpha_\beta, \alpha, \beta < \omega_\tau\}$, *where X_α are discrete spaces, in the case where the uniformity is complete, or X has a dense embedding into* $\varprojlim \{X_\alpha, \pi^\alpha_\beta \alpha, \beta < \omega_\tau\}$ *in the other case.*

Proof. Let $\{P_\alpha : \alpha \leqslant \omega_\tau\}$ be a l.o. base of weight $\tau > \chi_0$ consisting of coverings of order 1. We define $X_\alpha = P_\alpha$ where elements of P_α are treated as points of a discrete space. We define the maps $\pi^\alpha_\beta : X_\alpha \to X_\beta$ for $\alpha \geqslant \beta$, $\alpha, \beta \leqslant \omega_\tau$ as follows: $\pi^\alpha_\beta(u) = v$ iff $u \subset v$, where $u \in P_\alpha$, $v \in P_\beta$. The family $\{X_\alpha, \pi^\alpha_\beta, \alpha, \beta < \omega_\tau\}$ forms an inverse system having the desired properties.

THEOREM 5. *If a compact space has a uniformity with l.o. base, then it is metrizable.*

Proof. If X is a compact space, then there exists only one uniformity with a base consisting of all finite open coverings of X (see [1], p. 336) Supose that X is non-metrizable. We may assume that card $X \geqslant \chi_0$. We shall construct by induction a sequence $P_0 \prec P_1 \prec P_2 \prec \ldots$ of finite open coverings such that:

$$(*) \qquad \operatorname{card} P_0 \lessdot \operatorname{card} P_1 \lessdot \operatorname{card} P_2 \lessdot \ldots$$

Let $P_0 = \{X\}$. Let us assume that we have defined irreducible and finite open coverings P_i, for $i = 1, 2, \ldots, n-1$, such that card $P_0 \lessdot$ card $P_1 \lessdot \ldots$ $\lessdot$ card P_{n-1}

and
$$P_0 \prec P_1 \prec \ldots \prec P_{n-1}$$

There exist two points $x, y \in X$ such that $y \in st\ (x, P_{n-1})$. Let us take open and irreducible covering P_n such that $P_n \succ P_{n-1}$ and $y \notin st\ (x, P_n)$. It is easy to verify that card $P_n >$ card P_{n-1}. From Lemma 1 it follows that there exists an open finite covering P such that $P \succ P_i$ for $i = 1,2,\ldots$ This implies that card $P \geqslant$ card P_i for $i = 1,2,\ldots$ Thus, according to (*), card P is infinite; a contradiction.

Now we shall give some examples of spaces having uniformity with l.o. base.

EXAMPLE 1. Let X be the set of ordinal numbers

$$X = \{1,2,\ldots, \omega_0, \ldots, \omega_1\}$$

For every $\alpha \leqslant \omega_1$ we define P_α as follows: P_α consists of one-point sets $\{\beta\}$ for $\beta < \alpha$ and the set $[\alpha, \omega_1] = \{\gamma : \alpha \leqq \gamma \leqq \omega_1\}$. Now let us notice that $P_\alpha \prec P_\beta$ for $\alpha \leqslant \beta$. Thus the coverings P_α, $\alpha < \omega_1$, form a l.o. base for a uniformity on X. The topology induced by this uniformity is non-metrizable, since the weight of the space X at the point ω_1 is greater than $\aleph_0$.

EXAMPLE 2. There exists a metrizable space which has a uniformity with non-metrizable l. o. base.

The space $Y = X \setminus \{\omega_1\}$ where X is the space from Example 1 is discrete, but the uniformity with l. o. base consisting of coverings of the form $R_\alpha = P_\alpha \wedge Y$, $\alpha < \omega_1$, where $P_\alpha \wedge Y = \{u \cap Y : u \in P_\alpha\}$ is not metrizable.

EXAMPLE 3. We shall give here an example of a non-metrizable space without isolated points having a uniformity with l. o. base

Let the space X be the inverse limit of a system S consisting of discrete spaces X_α and maps $\pi^\alpha_\beta : X_\alpha \to X_\beta$, for $\alpha, \beta < \omega_1$, defined by induction as follows:

1. $X_1 = \{0\}$.
2. Let us assume that we have defined discrete spaces X_α, and maps $\pi^\alpha_\beta : X_\alpha \to X_\beta$, $\alpha \geqslant \beta$, satisfying $\pi^\beta_\gamma \pi^\alpha_\beta = \pi^\alpha_\gamma$, $\alpha \geqslant \beta \geqslant \gamma$, for all α, β, $\gamma < \lambda$, where $\lambda < \omega_1$.

a) If the number λ is the succesor of a number α, i.e. $\alpha + 1 = \lambda$, then we define X_λ to be the discrete space $X_\alpha \times \{1, 2\}$, and the map $\pi^\alpha_\beta : X_\alpha \to X_\beta$ to be such that $\pi^\lambda_\alpha (x, i) = x$ for $(x, i) \in X_\lambda$ and π^λ_β for $\beta \leqslant \lambda$ to be the composition $\pi^\alpha_\beta \pi^\lambda_\alpha$.

b) If λ is a limit number, then let $Z_\lambda = \varprojlim \{X_\alpha, \pi^\alpha_\beta, \alpha < \beta < \lambda\}$ and let $\pi_\alpha : Z_\lambda \to X_\alpha$ be the projection onto X_α. We define X_λ to be the set $Z_\lambda \times \{1, 2\}$ with the discrete topology, $\pi^\lambda_\alpha : X_\lambda \to X_\alpha$ for $\alpha \leqslant \lambda$ to be the composition $\pi_\alpha \pi^\lambda$, where $\pi^\lambda : X_\lambda \to Z_\lambda$, $\pi^\lambda (x, i) = x$ for $x \in Z_\lambda$, $i = 1, 2$.

Thus the system S is defined.

The space $X = \overleftarrow{\lim} S$ has a uniformity with non-metrizable l. o. base consisting of coverings P_α, $\alpha < \omega_1$, where $P_\alpha = \{\pi_\alpha^{-1}(x_\alpha) : x_\alpha \in X_\alpha\}$, where $\pi_\alpha : X \to X_\alpha$ are projections.

From the fact that X_α are discrete and from formula $\pi_\beta = \pi_\beta^\alpha \pi_\alpha$ follows that $P_\alpha \succ_* P_\beta$ for $\alpha > \beta$. The elements of topological base on X are sets of the form $\pi_\alpha^{-1}(x_\alpha)$, $x_\alpha \in X_\alpha$, $\alpha < \omega_1$. Since counterimages by π_β^α of every point of X_β have at least two points, every open basic set $\pi_\alpha^{-1}(x_\alpha)$ in X contains at least two points.

Thus all points in X are non-isolated.

EXAMPLE 4. There exists a non-metrizable space X with ind $X = 0$, having no uniformity with l. o. base.

Let W be the space of rational numbers and let C be the subset of W consisting of all integers. The quotient space $X = W/C$ has ind $X = 0$ but is not discrete. It is not metrizable, since it has no countable base at the point C. But every point $x \in X$ is a G_δ set. So if X had a uniformity with l. o. base, then the base could not be countable and every one-point set from X, being a G_δ set, would be open. But X is not discrete, and we get a contradiction.

Let us note that if X and Y are spaces with l. o. bases, then $X + Y$ (disjoint union of X and Y) and $X \times Y$ are not necessarily spaces with l. o. bases.

EXAMPLE 5. Let $Z = [0, 1] + X$, where X is the space from Example 1. Spaces $[0, 1]$ and X have uniformities with l. o. bases. Suppose that the space Z has a uniformity with l. o. base B. The space X is not metrizable so dim $Z = 0$; a contradiction with dim $Z = 1$.

EXAMPLE 6. Let $Z = X \times [0, 1]$, where X is from Example 1. Space Z is not metrizable, so if Z has a uniformity with l. o. base, then the base must be such that every G_δ set is open in Z. Let $\pi : X \times [0, 1] \to [0, 1]$ be the projection. Since the set $\pi^{-1}(0) = \bigcap_{i=1}^{\infty} \pi^{-1}([0, \frac{1}{i}))$ is G_δ in Z, it is open in Z. Since π is an open map, $\pi(\pi^{-1}(0)) = \{0\}$ is open in $[0, 1]$; a contradiction.

It is easy to prove that if F is a family of spaces with l. o. bases having the same ordinal type, then the disjoint union of F is a space with l. o. base. We do not know whether this is true for the product of F, if F is infinite.

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PRZESTRZENIE O JEDNOSTAJNOŚCIACH Z BAZĄ LINIOWO UPORZĄDKOWANĄ

Streszczenie

W pracy badamy własności topologii wyznaczonych przez jednostajności o bazie liniowo uporządkowanej przez wpisywanie gwiaździste pokryć.

Dowodzimy m. in., że topologia wyznaczona przez jednostajność z bazą liniowo uporządkowaną jest parazwarta; jeśli jest zwarta, to jest metryzowalna; jeśli nie jest metryzowalna, to jest zerowymiarowa i jest podprzestrzenią granicy odwrotnej układu dobrze uporządkowanego przestrzeni dyskretnych.

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